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Bayesian evidence for non-zero θ_{13} and CP-violation in neutrino oscillations

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ABSTRACT: We present the Bayesian method for evaluating the evidence for a non-zero value of the leptonic mixing angle θ_{13} and CP-violation in neutrino oscillation experiments. This is an application of the well-established method of Bayesian model selection, of which we give a concise and pedagogical overview. When comparing the hypothesis $\theta_{13} = 0$ with hypotheses where $\theta_{13} > 0$ using global data but excluding the recent reactor measurements, we obtain only a weak preference for a non-zero θ_{13}, even though the significance is over 3σ. We then add the reactor measurements one by one and show how the evidence for $\theta_{13} > 0$ quickly increases. When including the DOUBLE CHOOZ, DAYA BAY, and RENO data, the evidence becomes overwhelming with a posterior probability of the hypothesis $\theta_{13} = 0$ below 10^{-11}. Owing to the small amount of information on the CP-phase δ, very similar evidences are obtained for the CP-conserving and CP-violating hypotheses. Hence, there is, not unexpectedly, neither evidence for nor against leptonic CP-violation. However, when future experiments aiming to search for CP-violation have started taking data, this question will be of great importance and the method described here can be used as an important complement to standard analyses.

Contents

1   Introduction

The phenomenon of neutrino oscillations has been established in a series of experiments using neutrinos from a wide range of sources, such as the Earth's atmosphere, the Sun, nuclear reactors, and man-made accelerators [1–3]. Neutrinos can only oscillate if they are massive, and there are many important questions raised related to neutrino masses, which are currently being investigated. These include the Dirac or Majorana nature of neutrino masses, and the question of whether total lepton-number is violated, information on which can be obtained through experiments searching for neutrinoless double beta decay [4–6]. In addition, there is the quest for the determination of the absolute values of the neutrino masses, which is most directly performed using single beta decay experiments [7, 8], although cosmological observations can also provide information [9].

However, there are still many questions related to oscillation experiments which remain to be answered. The oscillations observed until recently correspond to the dominant effective two-flavor oscillation modes, driven by two mass-squared differences and two relatively large mixing angles, while the purpose of most current and future experiments is mainly to search for sub-leading effects. This includes the determination of the third leptonic mixing angle θ_{13} and of the neutrino mass ordering, which can either be normal or inverted. Also, the existence of CP-violation in the lepton sector is a very important question.

In fact, CP-violation in neutrino oscillations, which is a genuine three-flavour effect, is only possible for a non-zero value of θ_{13}, and any realistic possibility to determine the type of the neutrino mass ordering relies on θ_{13} not being too small [10]. Therefore, the results on θ_{13} from the present generation of experiments will be of crucial importance for the feasibility of future experiments aiming to determine the neutrino mass ordering or search for leptonic CP-violation.

For a long time, the neutrino oscillation data was largely consistent with a zero value of θ_{13}. However, a while ago, there was a series of experiments indicating a preference for a non-zero value of θ_{13} from the MINOS [11], T2K [12], and DOUBLE CHOOZ [13] collaborations. These measurements were recently followed by much more precise ones from DAYA BAY [14] and RENO [15].

Due to the degeneracies between oscillation parameters in individual neutrino oscillation experiments, there exists a long history of global fits of oscillation data, with some of the most recent ones being Refs. [1–3]. These analyses apply standard frequentist methods in order to estimate the oscillation parameters and to determine the significance of $\theta_{13} > 0$ (see also Refs. [16, 17]), but do not include any of the recent reactor data of the DOUBLE CHOOZ, DAYA BAY, and RENO collaborations. Bayesian estimation of neutrino oscillation parameters using subsets of neutrino oscillation data has been performed in Refs. [18, 19]. Finally, after the completion of this work, an update of Refs. [1, 2] was made public, including the recent reactor data [20].

In this work, we describe how to correctly determine the evidence for a non-zero value of the leptonic mixing angle θ_{13} and a non-trivial value of the CP-violating Dirac phase δ using Bayesian inference. The well-established method of analyzing these types of questions is based on *Bayesian model selection*, which we will describe in some detail. More information on model selection can be found in the textbooks and review articles in Refs. [21–25]. Applications within the fields of cosmology and astrophysics can be found in Refs. [21, 23, 24, 26–28], while applications in particle physics are given in Refs. [29–33]. We will compare the hypothesis $\theta_{13} = 0$ with the hypotheses where θ_{13} can take on any value within its physical range, and in this way obtain the preference of a non-zero value of θ_{13}. After assigning prior probabilities on the different hypotheses, we obtain posterior probabilities of the hypotheses that $\theta_{13} > 0$ and $\theta_{13} = 0$, respectively. We evaluate how recent data has increased step by step the evidence so that it now overwhelmingly prefers a non-zero θ_{13} (which is not very surprising, given the recent reactor measurements). Furthermore, we automatically obtain the method of evaluating the Bayesian evidence for a non-trivial value of the phase δ. Since current data is not very sensitive to the value of δ, we find, not surprisingly, no evidence neither for nor against CP-violation in neutrino oscillations. However, in the future when proposed experiments aiming to search for CP-violation (see, for example, Refs. [34–37]) have started giving data, this question will be of great importance, and the method described here can be used to assess to which degree that data favors or disfavors CP-violation in neutrino oscillations.

This work is organized as follows. In Sec. 2, we review the principles of Bayesian inference, including a self-contained treatment of Bayesian model selection. Sec. 3 describes our method of performing model selection for determining the evidence favoring a non-zero θ_{13} and leptonic CP-violation, and includes model definitions, a thorough discussion on the choice of priors, and the approximation of the likelihood we need to make. The numerical results are presented, first using the global data analyzed in Ref. [2], and then also including the recent DOUBLE CHOOZ, DAYA BAY, and RENO data. Finally, a brief summary and our conclusions are given in Sec. 4.

2 Bayesian inference

In the Bayesian interpretation, probability is associated with degree of belief. This is in contrast to the frequentist interpretation, in which probability is defined as the limit of the relative frequency of an event in a large number of repeated trials.

Bayesian inference is a framework for updating prior belief or knowledge based on new information or data. A common problem in data analysis is to use the data to make inferences about parameters of a given model, and at a higher level, to decide which of two or more competing models is preferred by the data. Bayesian inference answers the question how probable a given value of a parameter, or a whole model, is, given the observed data. If one is interested in probabilities of parameters within models, as well as the models themselves, one is forced to adopt the Bayesian approach. Generally, the probability $\Pr(A|B)$ represents the degree of belief regarding the truth of A, given B. The order of the conditioning can be reversed using *Bayes' theorem*,

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}. \tag{2.1}$$

Often, one is interested to infer values of parameters of a model from a set of observations or data. Given a model or hypothesis H with a set of N parameters $\boldsymbol{\Theta} = \{\Theta_i\}_{i=1}^{N}$, and a set of data $\mathbf{D} = \{D_i\}_{i=1}^{M}$, Bayes' theorem implies[1]

$$\Pr(\boldsymbol{\Theta}|\mathbf{D}, H) = \frac{\Pr(\mathbf{D}|\boldsymbol{\Theta}, H)\Pr(\boldsymbol{\Theta}|H)}{\Pr(\mathbf{D}|H)}, \tag{2.2}$$

where $\Pr(\boldsymbol{\Theta}|\mathbf{D}, H)$ is the posterior probability (density) of the parameters $\boldsymbol{\Theta}$, given the model and the data, and $\pi(\boldsymbol{\Theta}) \equiv \Pr(\boldsymbol{\Theta}|H)$ is the prior probability (density). The likelihood function $\mathcal{L}(\boldsymbol{\Theta}) \equiv \Pr(\mathbf{D}|\boldsymbol{\Theta}, H)$ is the probability (density) of the data $\mathbf{D}$, assuming parameter values $\boldsymbol{\Theta}$, while $\Pr(\mathbf{D}|H)$ is the *Bayesian evidence* (or model likelihood), which is given by

$$\mathcal{Z} \equiv \Pr(\mathbf{D}|H) = \int \Pr(\mathbf{D}, \boldsymbol{\Theta}|H)\mathrm{d}^N\boldsymbol{\Theta} = \int \Pr(\mathbf{D}|\boldsymbol{\Theta}, H)\Pr(\boldsymbol{\Theta}|H)\mathrm{d}^N\boldsymbol{\Theta}$$
$$= \int \mathcal{L}(\boldsymbol{\Theta})\pi(\boldsymbol{\Theta})\mathrm{d}^N\boldsymbol{\Theta}, \tag{2.3}$$

and is simply the factor required to normalize the posterior in Eq. (2.2). Since the evidence does not depend on the values of the parameters $\boldsymbol{\Theta}$, it is usually ignored in parameter estimation problems and the parameter values and uncertainties are obtained using the unnormalized posterior. However, the evidence plays a central role in model selection, as will be described in Sec. 2.1.

One important thing to keep in mind is that prior and posterior probability densities, in order to keep the total probability invariant, transform under a change of variables $\boldsymbol{\Theta} \to \boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{\Theta})$ by multiplication by the Jacobian determinant, *i.e.*, as

$$\Pr(\boldsymbol{\Omega}) = \Pr(\boldsymbol{\Theta})\left|\frac{\partial\boldsymbol{\Theta}}{\partial\boldsymbol{\Omega}}\right|. \tag{2.4}$$

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[1] All probabilities are also implicitly assumed to be conditioned on all the relevant background information I, *i.e.*, $\Pr(X)$ is written instead of $\Pr(X|I)$.

Hence, a prior uniform in one parameter will not be so in a nonlinear function of it, and thus the specification of a prior is essentially equivalent to the specification of a variable in which the prior is uniform.

The probability density of any subset η of the parameters $\boldsymbol{\Theta} = (\eta, \rho)$ is obtained by integrating over the other parameters ρ, fully taking into account their uncertainty, as

$$\Pr(\eta|X) = \int \Pr(\eta, \rho|X)\mathrm{d}\rho, \tag{2.5}$$

for any X. This makes it possible to eliminate nuisance parameters in a fully consistent way by including them in the parameter space and then performing the above integral over the posterior distribution. In addition, the probability density of any (unique) function of the parameters $K = F(\boldsymbol{\Theta})$ is obtained as

$$\Pr(K|X) = \int \Pr(K|\boldsymbol{\Theta}, X) \Pr(\boldsymbol{\Theta}|X)\mathrm{d}^N\boldsymbol{\Theta} = \int \delta(K - F(\boldsymbol{\Theta})) \Pr(\boldsymbol{\Theta}|X)\mathrm{d}^N\boldsymbol{\Theta}. \tag{2.6}$$

Although this might look like a daunting integral, if one has access to samples from $\Pr(\boldsymbol{\Theta}|X)$, one can easily find the total probability in an interval of K by simply binning the samples.

The main result of Bayesian parameter inference is the posterior and its marginalized versions (usually in one or two dimensions). However, it is also common to give point estimates such as the posterior mean or median, as well as *credible intervals (regions)*, which are defined as intervals (regions) containing a certain amount of posterior probability. Note that these regions are not unique without further restrictions, just as for classical confidence intervals.

Examples of Bayesian analyses of particle physics models are global fits of supersymmetric extensions of the standard model [29, 30, 38–41] and of the minimal universal extra dimensions scenario [42].

2.1 Model selection

The discussion in the previous section was concerned with the Bayesian method of inferring the values of parameters, *assuming a certain hypothesis H to be true*. However, another arguably much more important question is that of which hypothesis is the best description of the data in the first place. It is the evidence in Eq. (2.3) which can be used to distinguish between a set of hypotheses $\{H_i\}_{i=1}^r$. This is because Bayes' theorem also implies

$$\Pr(H_i|\mathbf{D}) = \frac{\Pr(\mathbf{D}|H_i)\Pr(H_i)}{\Pr(\mathbf{D})}, \tag{2.7}$$

and so gives the posterior ratio of probabilities of two hypotheses as

$$\frac{\Pr(H_i|\mathbf{D})}{\Pr(H_j|\mathbf{D})} = \frac{\Pr(\mathbf{D}|H_i)}{\Pr(\mathbf{D}|H_j)}\frac{\Pr(H_i)}{\Pr(H_j)} = \frac{\mathcal{Z}_i}{\mathcal{Z}_j}\frac{\Pr(H_i)}{\Pr(H_j)}, \tag{2.8}$$

where $\Pr(H_i)/\Pr(H_j)$ is the prior probability ratio of the two models. The ratio of evidences, $B_{ij} = \mathcal{Z}_i/\mathcal{Z}_j$ is often called the *Bayes factor*. From Eq. (2.3), one observes that

$\log(\mathcal{Z}_1/\mathcal{Z}_0)$	$\mathcal{Z}_1/\mathcal{Z}_0$	$\Pr(H_1|\mathbf{D})$	Interpretation
< 1.0	$\lesssim 3:1$	$\lesssim 0.75$	Inconclusive
1.0	$\simeq 3:1$	$\simeq 0.75$	Weak evidence
2.5	$\simeq 12:1$	$\simeq 0.92$	Moderate evidence
5.0	$\simeq 150:1$	$\simeq 0.993$	Strong evidence

Table 1. Jeffrey's scale often used for the interpretation of Bayes factors and model probabilities. The posterior model probabilities are calculated by assuming only two competing hypotheses and equal prior probabilities.

the evidence is the average of the likelihood over the prior, and hence this method automatically implements a form of *Occam's razor*, since in general a simpler theory with a smaller parameter space will have a larger evidence than a more complicated one, unless the latter can fit the data substantially better. More specifically, only the inclusion of parameters which are constrained by the data will lead to a smaller evidence, while the inclusion of parameters that are unconstrained will leave the evidence unaffected. Since $\Pr(\mathbf{D}) = \sum_i \Pr(\mathbf{D}|H_i)\Pr(H_i)$, Eq. (2.7) can easily be written in terms of the evidences and the prior probabilities as

$$\Pr(H_j|\mathbf{D}) = \frac{1}{1 + \sum_{i \neq j} \frac{\mathcal{Z}_i}{\mathcal{Z}_j} \frac{\Pr(H_i)}{\Pr(H_j)}}. \tag{2.9}$$

In the simplest case when two models being compared have no free parameters, the Bayes factor is simply the likelihood ratio. When the models have free parameters, it is still a likelihood ratio, but between the whole models, and this ratio is obtained by integrating over the parameter spaces of the models as in Eq. (2.3). Note that Bayesian model selection allows data to favor the simpler model. Also, it is possible to incorporate external information when comparing models. In addition, note that the posterior distributions of the parameters do not depend on the overall scale of the likelihood, since the evidence scales accordingly. The same holds true for the posterior model probabilities, since the ratio of evidences in Eq. (2.9) is also independent of the likelihood normalization. The significance of the evidence is usually interpreted using *Jeffrey's scale* in Tab. 1, as used in, for example, Refs. [21, 23, 24, 29, 30].

Generally, as more data become available, for any (not completely unreasonable) prior distribution, the posterior becomes practically independent of the prior, and determined solely by the likelihood. However, the dependence of the evidence on the prior always remains, although the Bayes factor will generally favor the correct model once "enough" data has been obtained [23].

It can be interesting to compare this method with the completely different approach used in frequentist inference in the special case of nested models. In the latter case, the significance of a new effect is usually evaluated using hypothesis tests [43]. The null hypothesis H_0 of no effect is expressed as $\eta = \eta_0$ and is tested against the alternative hypothesis $\eta \neq \eta_0$. A significance level α is chosen and a test statistic with known probability distribution under H_0 is constructed. Using the observed data, an "observed" value of the test

statistic and a p-value p is calculated. The null hypothesis is rejected if $p < \alpha$.[2] Note that the p-value is not directly related to the probability of the null hypothesis being true.

A very commonly used statistic is based on the *profile likelihood ratio*. Define

$$Q^2(\mathbf{D}, \eta_0) \equiv -2 \log \frac{\sup_\rho \mathcal{L}(\eta_0, \rho)}{\sup_{\eta,\rho} \mathcal{L}(\eta, \rho)} = -2 \log \frac{\mathcal{L}(\eta_0, \hat{\hat{\rho}}(\eta_0))}{\mathcal{L}(\hat{\eta}, \hat{\rho})}, \qquad (2.10)$$

where "sup" denotes the supremum, a single hat denotes the parameters which maximize the likelihood, and a double hat indicates the conditional maximum for fixed η_0. Since in this case the dependence on the data $\mathbf{D}$ is important, it is explicitly shown as an argument. Under the assumption that H_0 is true, in the large sample limit and under some additional conditions, Q^2 (often denoted by $\Delta\chi^2$) has a χ^2-distribution with number of degrees of freedom equal to the dimensionality of η. This result is known as *Wilks' theorem* [44]. Frequentist *confidence intervals* at confidence level $1 - \alpha$ can then be constructed by performing the hypothesis test for all values of the parameter and then including all parameter values that are not rejected at a significance α. This close relationship between between interval construction and hypothesis testing does not, however, have any analogy in Bayesian inference, *i.e.*, there is no direct relation between parameter estimation and credible interval construction on the one hand and model selection on the other hand. Note that a more complicated model will always be able to fit the data at least as good as the simpler model, and hence, unlike in Bayesian model selection, one can never obtain evidence *in favor* of the null hypothesis.

Sometimes it may happen that a conclusion based on a hypothesis test seems to contradict that obtained using model selection. This can, for example, happen if the estimate of η is found far from the value of the null hypothesis and a large value of Q^2 is observed, and so the data is unlikely under H_0. However, the data could be even more unlikely under the alternative as in Eq. (2.3), and hence the Bayes factor can even favor H_0. Also, the prior probability of the alternative could be very low. Thus, taking the Bayesian view also means that the significance, or the "number of σ", of a result is in general not a good indicator of the importance or the evidence of a new effect, a result that is known as "Lindley's paradox". For further details, see for example Appendix A of Ref. [23]. As we will see, this situation does, to some extent, apply to the case of the third leptonic mixing angle θ_{13} when the recent reactor measurements are not taken into account.

We want to emphasize that the posterior *probability density* at the special value $\eta = \eta_0$ of the alternative model says nothing about the probability that $\eta = \eta_0$, and neither does the relative posterior densities at $\eta = \eta_0$ and the maximum of the posterior. The reason is that, if one uses a continuous prior probability density for η, then the probability that $\eta = \eta_0$ is zero for any value of η_0. However, for nested models and under some additional assumptions, the ratio of the prior to the posterior at $\eta = \eta_0$ does indicate this probability, since it is in fact the Bayes factor [23].

Finally, we mention that the Bayesian analysis can be taken even one step further by performing *model averaging*. Using this, the posterior for any set of parameters can be

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[2]The p-value is not the significance of the test, but it is the highest significance at which H_0 could be rejected. This distinction will be implicitly assumed for the rest of this work.

calculated, while taking into account the uncertainty regarding which model is the correct one. Once again, all one has to do is to use the laws of probability theory to obtain [21, 24]

$$\Pr(\boldsymbol{\Theta}|X) = \sum_{i=1}^{r} \Pr(\boldsymbol{\Theta}|H_i, X) \Pr(H_i|X), \tag{2.11}$$

i.e., the probability distribution is given by the average of the individual distributions over the space of models, with weights equal to the model probabilities.

3   Bayesian model selection for θ_{13} and δ

3.1   Model definitions

As the general framework, we simply take the Standard Model and augment it with a neutrino mass matrix. The nature of the mass matrix (Dirac or Majorana) is not important here, but we observe that the weakly interacting neutrino fields ν_α ($\alpha = e, \mu, \tau$) are superpositions of mass eigenstate fields ν_i ($i = 1, 2, 3$) with masses m_i. The neutrino masses m_i can either have normal ($m_1 < m_2 < m_3$) or inverted ($m_3 < m_1 < m_2$) ordering. In a basis where the charged lepton mass matrix is diagonal we have

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i} \nu_i, \tag{3.1}$$

where U is the *leptonic mixing matrix*, also referred to as the *Pontecorvo-Maki-Nakagawa-Sakata* (PMNS) matrix, usually parametrized as

$$
\begin{aligned}
U &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{diag}\left(e^{i\rho}, e^{i\sigma}, 1\right) \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{3.2}
$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, θ_{12}, θ_{23}, and θ_{13} are the lepton mixing angles, δ is the CP-violating Dirac phase, and σ and ρ are CP-violating Majorana phases, which are only relevant in the case of Majorana neutrinos. The physical ranges for the mixing angles are the intervals $[0, \pi/2]$ [45], while δ can be restricted to $[-\pi, \pi]$, and the Majorana phases to $[0, \pi]$. With $\Delta m_{21}^2 = m_2^2 - m_1^2$ and $\Delta m_{31}^2 = m_3^2 - m_1^2$, neutrino oscillation experiments are sensitive to the set of parameters $\boldsymbol{\Theta}_{\mathrm{osc}} = (\Delta m_{21}^2, \Delta m_{31}^2, \theta_{12}, \theta_{23}, \theta_{13}, \delta)$, meaning that these are the parameters that the oscillation probabilities depend on. Defining

$$\psi = \left(\Delta m_{21}^2, \Delta m_{31}^2, \theta_{12}, \theta_{23}\right), \tag{3.3}$$

we distinguish the two CP-conserving cases $\delta = 0$ and $\delta = \pi$, and define the following hypotheses

$$H_0 : \boldsymbol{\Theta}_0 = \psi, \ \theta_{13} = 0 \quad (\delta \text{ irrelevant}), \tag{3.4}$$

$$H_1 : \boldsymbol{\Theta}_1 = (\psi, \theta_{13}), \ \theta_{13} \in [0, \pi/2], \ \delta = 0, \tag{3.5}$$

$$H_2 : \boldsymbol{\Theta}_2 = (\psi, \theta_{13}), \ \theta_{13} \in [0, \pi/2], \ \delta = \pi \quad (\text{ or } -\pi), \tag{3.6}$$

$$H_3 : \boldsymbol{\Theta}_3 = (\psi, \theta_{13}, \delta), \ \theta_{13} \in [0, \pi/2], \ \delta \in [-\pi, \pi]. \tag{3.7}$$

These hypotheses can then be compared by computing the evidences in Eq. (2.3). For this one requires the likelihood function and priors on the oscillation parameters. These will be described in Secs. 3.3 and 3.2, respectively.

The neutrino mass ordering will be treated as fixed, and the very small differences obtained by assuming the different orderings to be true will be evaluated in Sec. 3.4. However, once more data from future experiments searching for CP-violation become available, the constraints on the phase δ is expected to depend very much on the assumed mass ordering [46]. In this case, it might be preferable to define separate models for each mass ordering, and perform model selection on all eight models simultaneously.

3.2 Choice of priors

In any Bayesian analysis, one needs to specify prior probability distributions on the parameters in a given model, and if more than one model is considered, probabilities on the space of models. In general, the prior should reflect ones prior knowledge, given the relevant background information. However, some difficulties can appear if there is very little background information, or if it is difficult to translate this information into mathematically precise statements.

First, we consider the question of how to assign prior model probabilities. A reasonable choice could be to use prior probabilities $\Pr(H_0) = 1/3$, $\Pr(H_1) = \Pr(H_2) = 1/6$, and $\Pr(H_3) = 1/3$. Then, however, the a priori probability of $\theta_{13} = 0$ would only be $1/3$ rather than $1/2$, which could be considered more natural. This will, as always, only have a non-negligible effect on the posterior probabilities (as it should) when the data is not very informative. Even in this case the effect will be small, and actually much smaller than the effect coming from the variations in the evidences when different priors of θ_{13} are chosen.

Then, there is the task of assigning prior probability densities on the continuous parameters of the models. To deal with this, a wide variety of methods and rules have been developed which one can use to obtain priors in cases when there is little prior information (see, for example, Refs. [47, 48]). These methods can serve as a guide to what shapes a reasonable distribution might have. However, we take the common point of view that, in general, these priors should not be accepted too blindly. In the end, the priors should be proper, *i.e.*, normalized to unity, and be a reasonable reflection of one's degree of belief. By considering the variation of the posterior inference when a set of different such priors is considered, one can check the robustness of the obtained results.

We assume a prior on the form

$$\pi(\psi, \theta_{13}, \delta) = \pi(\psi)\pi(\theta_{13})\pi(\delta) \tag{3.8}$$

on the oscillation parameters. Besides being quite natural, this form of the prior can be obtained by demanding that the prior must not depend on which basis the neutrino mass matrix is defined in. The resulting so-called *Haar measures* on the leptonic mixing matrix have been discussed in Refs. [49, 50],[3] which lead to unique and separable priors on the mixing angles and phases. The prior on ψ is taken to be the same for all hypotheses.

Our form of the evidences, derived in Sec. 3.3, will be independent of the prior on ψ, and hence from now on we only consider the priors on θ_{13} and δ. In principle, one could simply make up a list of different reasonable priors, or equivalently variables in which one chooses a uniform prior, to investigate. The different choices of priors on θ_{13} considered in this work are those derived from the Haar measures on SO(2), U(2), SO(3), and U(3), respectively.[4] Also, since the experiments most sensitive to θ_{13}, *i.e.*, those involving accelerator and reactor neutrinos, are mainly sensitive to $\sin^2(2\theta_{13})$, one can choose a prior uniform in $\sin^2(2\theta_{13})$, in which case Eq. (2.4) is used to extend the prior to the whole physical range $\theta_{13} \in [0, \pi/2]$. The different priors can be summarized as

$$
\begin{aligned}
\mathcal{A}: & \quad \pi(\theta_{13}) = 2/\pi \\
\mathcal{B}: & \quad \pi(s_{13}^2) = 1 \\
\mathcal{C}: & \quad \pi(s_{13}) = 1 \\
\mathcal{D}: & \quad \pi(c_{13}^4) = 1 \\
\mathcal{E}: & \quad \pi\left(\sin^2(2\theta_{13})\right) = 1,
\end{aligned}
\tag{3.9}
$$

where $\theta_{13} \in [0, \pi/2]$ and all other variables are in the interval $[0, 1]$. The corresponding implied priors on θ_{13} according to Eq. (2.4) are plotted in Fig. 1, where also the approximate region in which most of the contribution to the evidences originates from is marked with a grey band. It is important to note that we do not consider any of the studied priors to be "better" than the others, but instead consider them all as reasonable, and then evaluate how much the posterior inferences depend on the choice of priors.

Note that, in the rough approximation of a box-shaped likelihood and a constant prior within this region, the evidence is proportional to that constant value of the prior. Hence, one can estimate how the evidence for models with non-zero θ_{13} will depend on the prior chosen. Most importantly, the prior $\mathcal{C}$ will give the largest evidence, approximately a factor of two to four larger than $\mathcal{B}$, which yields the smallest evidence. This will be checked numerically in Sec. 3.4.

For the model H_3, one also needs to specify a prior on δ. However, since the constraints on δ are so weak, not only will its inclusion in the first place effect the evidence very little, but also the form of its prior will be largely irrelevant. The Haar measure is uniform in δ, and so we use $\pi(\delta) = 1/(2\pi)$. If, in the future, the data becomes more informative regarding the value of δ, one would need to evaluate if that data indeed supports the existence of CP-violation in neutrino oscillations, and the general method presented in this work can be followed. In this case, one could also evaluate the sensitivity to the choice of the prior

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[3]However, note the arising complications due to the unphysical phases discussed in Ref. [50].

[4]That is, corresponding to real two-flavor, complex two-flavor, real three-flavor, and complex three-flavor mixing, respectively.
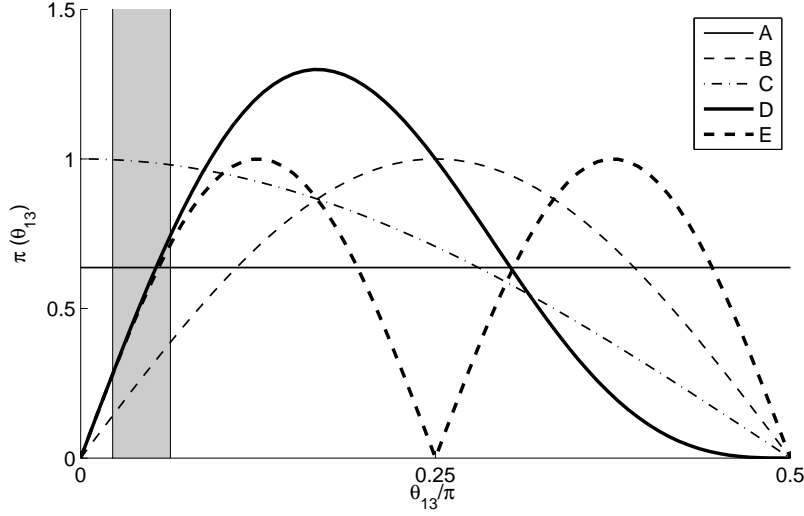
Figure 1. The different correctly normalized priors on θ_{13}. The grey band marks the approximate region from which the main contribution to the evidence originates.

on δ. Since it is $\sin\delta$ and $\cos\delta$ which appears in the leptonic mixing matrix in Eq. (3.2), and hence also enter into the oscillation probabilities and CP-asymmetries, suitable choices could be priors uniform in $\sin\delta$ and $\cos\delta$.

3.3 Approximate Bayes factors

Ideally, one would like to use the full likelihood of all the oscillation parameters, as well as the parameterization of the systematic uncertainties. However, since we do not have the machinery to do this, we are forced to make some simplifying assumptions. For any individual oscillation experiment, the likelihood as a function of all the oscillation parameters is in general highly degenerate and/or multi-modal. Nevertheless, thanks to the large number of different types of experiments, imposing constraints on different combinations of oscillation parameters, the likelihood has become unimodal and, at least to some approximation, Gaussian. The correlations between the different oscillation parameters in the standard parameterization are small, and the best-fit values of the different oscillation parameters are largely independent of the values of the other parameters [1–3]. We thus neglect any dependence between ψ as a whole and the pair (θ_{13}, δ) and approximate the likelihood as

$$\mathcal{L}(\psi, \theta_{13}, \delta) \simeq \mathcal{L}^{\mathrm{N}}(\psi)\mathcal{L}^{\mathrm{CPV}}(\theta_{13}, \delta). \tag{3.10}$$

With this approximation, the dependence of ψ factorizes out, and the two-dimensional marginalized posterior is simply

$$\Pr(\theta_{13}, \delta|\mathbf{D}, H_3) \propto \mathcal{L}^{\mathrm{CPV}}(\theta_{13}, \delta)\pi(\theta_{13}, \delta). \tag{3.11}$$

Furthermore, from Eq. (2.10), one easily obtains

$$Q^2(\theta_{13}, \delta) = -2\log\left(\frac{\mathcal{L}^{\mathrm{CPV}}(\theta_{13}, \delta)}{\mathcal{L}^{\mathrm{CPV}}(\hat{\theta}_{13}, \hat{\delta})}\right), \tag{3.12}$$

giving

$$\mathcal{L}^{\mathrm{CPV}}(\theta_{13}, \delta) = \mathcal{L}^{\mathrm{CPV}}(\hat{\theta}_{13}, \hat{\delta}) \exp\left(-\frac{Q^2(\theta_{13}, \delta)}{2}\right) \tag{3.13}$$

since the conditional maximum with respect to ψ is independent of θ_{13} and δ in this case. The quantity $\mathcal{L}^{\mathrm{CPV}}(\hat{\theta}_{13}, \hat{\delta})$ is an irrelevant factor which can be set to unity. The likelihood $\mathcal{L}^{\mathrm{CPV}}$ without recent reactor data (we denote this data set by GF) is evaluated using Eq. (3.13), and a map of Q^2 in the $\theta_{13} - \delta$ plane obtained from the authors of Ref. [2].[5]

Since the numbers of events in the reactor experiments DOUBLE CHOOZ (DC), DAYA BAY (DB), and RENO are quite large [13–15], the likelihoods are taken as Gaussian functions in the mean number of events, or equivalently, in $\sin^2(2\theta_{13})$,

$$\sin^2(2\theta_{13}) = 0.086 \pm 0.0508 \quad \text{(DC)}, \tag{3.14}$$

$$\sin^2(2\theta_{13}) = 0.092 \pm 0.0177 \quad \text{(DB)}, \tag{3.15}$$

$$\sin^2(2\theta_{13}) = 0.113 \pm 0.0230 \quad \text{(RENO)}, \tag{3.16}$$

where the systematic errors[6] have been integrated over using Gaussian distributions, giving the resulting effective likelihoods as Gaussians with unchanged means and deviation parameters added in quadrature.

From Eq. (2.3) and with the approximation in Eq. (3.10), one finds

$$\begin{aligned}
\mathcal{Z}_0 &= \int \mathcal{L}(\psi, 0, 0)\pi(\psi)\mathrm{d}\psi \\
&\simeq \int \mathcal{L}^{\mathrm{N}}(\psi)\pi(\psi)\mathrm{d}\psi \cdot \mathcal{L}^{\mathrm{CPV}}(0, 0),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\mathcal{Z}_3 &= \int \mathcal{L}(\psi, \theta_{13}, \delta)\pi(\psi, \theta_{13}, \delta)\mathrm{d}\psi\mathrm{d}\theta_{13}\mathrm{d}\delta \\
&\simeq \int \mathcal{L}^{\mathrm{N}}(\psi)\pi(\psi)\mathrm{d}\psi \cdot \int \mathcal{L}^{\mathrm{CPV}}(\theta_{13}, \delta)\pi(\theta_{13}, \delta)\mathrm{d}\theta_{13}\mathrm{d}\delta,
\end{aligned} \tag{3.18}$$

so that the ratio of the evidences is given by

$$\frac{\mathcal{Z}_3}{\mathcal{Z}_0} \simeq \frac{\int \mathcal{L}^{\mathrm{CPV}}(\theta_{13}, \delta)\pi(\theta_{13}, \delta)\mathrm{d}\theta_{13}\mathrm{d}\delta}{\mathcal{L}^{\mathrm{CPV}}(0, 0)}. \tag{3.19}$$

In a similar way, one obtains

$$\frac{\mathcal{Z}_1}{\mathcal{Z}_0} \simeq \frac{\int \mathcal{L}^{\mathrm{CPV}}(\theta_{13}, 0)\pi(\theta_{13})\mathrm{d}\theta_{13}}{\mathcal{L}^{\mathrm{CPV}}(0, 0)}, \tag{3.20}$$

$$\frac{\mathcal{Z}_2}{\mathcal{Z}_0} \simeq \frac{\int \mathcal{L}^{\mathrm{CPV}}(\theta_{13}, \pi)\pi(\theta_{13})\mathrm{d}\theta_{13}}{\mathcal{L}^{\mathrm{CPV}}(0, 0)}. \tag{3.21}$$

Although the approximation in Eq. (3.10) certainly introduces some error on the calculated evidences, it is expected to be much smaller than the uncertainty coming from the choice of prior distributions, which is often relatively large.

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[5]This corresponds to the "recommended" analysis of Ref. [2], which includes the short-baseline reactor data.

[6]We assume the systematic errors to be independent. Although this is not really correct, it should be a sufficiently good approximation for our purposes.

The required integrals can, due to the simple form of the likelihood $\mathcal{L}^{\mathrm{CPV}}$ and the low dimensionality, be evaluated using standard integration routines. For more complicated evidence integrals, more specific algorithms such as MULTINEST [51, 52], which we also use in this work, are required.

3.4 Numerical results

Our reconstructed two-dimensional posteriors using the data analyzed in Ref. [2], *assuming H_3 to be true*, and using prior $\mathcal{E}$, are shown in Fig. 2 for normal (left) and inverted (right) mass orderings. The 68 % and 95 % equal-posterior credible regions are marked by the black contours. Using different priors on θ_{13} results in very similar posteriors. In Fig. 3, the posteriors including all the recent reactor data are shown. Besides the obvious observation that now $\sin^2(2\theta_{13})$ is much better constrained, one notices that the preferred values of δ have changed. Although the added reactor data is completely insensitive to the value of δ, the degeneracy between δ and $\sin^2(2\theta_{13})$ in the other data results in changes of the preferred values. The maximum likelihood estimates are now $\hat{\delta} \simeq \pi$ and $\hat{\delta} \simeq 0$ for normal and inverted mass orderings, respectively. The one-dimensional marginalized posteriors are shown in Fig. 4. Our obtained maximum likelihood estimates and 68 % confidence intervals of $\sin^2(2\theta_{13})$ when using all available data are $\sin^2(2\theta_{13}) = 0.090 \pm 0.012$ and $\sin^2(2\theta_{13}) = 0.092 \pm 0.013$ for normal and inverted mass ordering, respectively.[7] Evaluating the posterior means and 68 % credible intervals yields essentially the same values.

Note that the peaks of these posteriors, which are obtained *assuming that θ_{13} is non-zero*, are quite far from $\theta_{13} = 0$, but in order to quantify the evidence of a non-zero θ_{13} one has to perform model selection. Due to the small amount of information regarding the value of δ, the evidences for the models H_1, H_2, and H_3 are very similar, $|\log(\mathcal{Z}_{2,3}/\mathcal{Z}_1)| \lesssim 0.2$ without DAYA BAY and RENO data, and $|\log(\mathcal{Z}_{2,3}/\mathcal{Z}_1)| \lesssim 0.5$ when including all data.[8] The difference in the logarithms of the Bayes factors when assuming the different mass orderings to be true (around 0.2 in log evidence) can be used as an estimate of the error on these logarithms induced by assuming the wrong mass ordering.

First, the analysis is done using only the data analyzed in Ref. [2]. After that, we calculate how the evidence for $\theta_{13} > 0$ changes when the DOUBLE CHOOZ, DAYA BAY, and RENO result are added, respectively. In Tab. 2, the evidences $\mathcal{Z} \simeq \mathcal{Z}_1 \simeq \mathcal{Z}_2 \simeq \mathcal{Z}_3$ compared to $\mathcal{Z}_0$ are shown. The ranges given are those obtained using the different priors on θ_{13}, but also the smaller variations coming from assuming different mass orderings are included.[9] Furthermore, the posterior probability of H_0 is shown, calculated assuming that $\Pr(H_0) = 1/2$ and $\Pr(H_0) = 1/3$ (in the parentheses), respectively. Note that, due to the similarity of the evidences $\mathcal{Z}_1, \mathcal{Z}_2$, and $\mathcal{Z}_3$ of the other models, the posterior probability of H_0 is independent of the prior probabilities of these models for fixed $\Pr(H_0)$. In the last column, the square root of the test statistic Q^2 is given for testing H_0 against H_1 (but with

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[7]The more careful analysis of Ref. [20] do not use the short-baseline reactor data and finds somewhat larger preferred values of θ_{13}.

[8]This does not depend on the prior on θ_{13}.

[9]The numerical errors on all logarithms of Bayes factors in this work are about 0.05 or smaller and can hence safely be neglected.

Data used	$\log(\mathcal{Z}/\mathcal{Z}_0)$	$\mathcal{Z}/\mathcal{Z}_0$	$\Pr(H_0|\mathbf{D})$	$\sqrt{Q^2}$
GF	$0.4 - 2.1$	$1.6 - 7.9$	$0.11(0.06) - 0.39(0.24)$	3.1
GF + DC	$1.7 - 3.1$	$5 - 22$	$0.04(0.02) - 0.15(0.08)$	3.5
GF + DC + DB	$14.2 - 15.4$	$(1.5 - 5) \cdot 10^6$	$< 7 \cdot 10^{-7}$	6.1
All	$25.3 - 26.5$	$(1 - 3) \cdot 10^{11}$	$< 10^{-11}$	7.8
Only DB	$9.1 - 10.3$	$(9 - 30) \cdot 10^3$	$< 10^{-4}$	5.2

Table 2. Summary of Bayes factors and posterior probabilities of H_0 for $\Pr(H_0) = 1/2$ (and $\Pr(H_0) = 1/3$ within parenthesis). The ranges of $\mathcal{Z} \simeq \mathcal{Z}_1 \simeq \mathcal{Z}_2 \simeq \mathcal{Z}_3$ given are those obtained using the different priors on θ_{13}, but also the smaller variation coming from the assumed mass ordering is included. The last column gives the square root of the observed Q^2 (only H_0 against H_1), which would equal the "number of σ" under the conditions of Wilks' theorem.

very small differences if testing against the other models), which would equal the "number of σ" under the conditions of Wilks' theorem.

First, we only use the data included in the global fit of Ref. [2] (again denoted by GF). Although there is more than a 3σ significance, there is in fact only weak evidence of $\theta_{13} > 0$, with logarithms of the Bayes factors lying in the range $0.4 - 2.1$ for the different priors on θ_{13}. The posterior probability of H_0, for this range of evidences, is in the range $0.11 - 0.39$ if $\Pr(H_0) = 1/2$ and in the range $0.06 - 0.24$ if $\Pr(H_0) = 1/3$. One can then also include the variation of the posterior probability of H_0 coming from the variation of $\Pr(H_0)$ between $1/3$ and $1/2$ to obtain the range $0.06 - 0.39$ for the posterior of H_0, including all discussed uncertainties. As discussed in Sec. 3.2, the prior $\mathcal{B}$ yields the smallest evidence due to the fact that it puts very little prior probability into the high-likelihood region (see Fig. 1).

When adding the DOUBLE CHOOZ measurements, the evidence for a non-zero θ_{13} increases. However, there is still no conclusive evidence. All logarithms of Bayes factors increase by about 1 or slightly more for the different priors, yielding the range $1.7 - 3.1$ for $\log(\mathcal{Z}/\mathcal{Z}_0)$ and posterior probabilities of H_0 (when including all uncertainties as in the previous paragraph) in the range $0.02 - 0.15$. The significance found is about 3.5σ.

Only the DAYA BAY data by itself yields a significance of 5.2σ, and also the Bayesian evidence strongly prefers a non-zero θ_{13}. The logarithms of Bayes factors are now all larger than 9, equivalent to posterior probabilities of H_0 smaller than 10^{-4}. When the DAYA BAY data is combined with the previous data, we find that $\log(\mathcal{Z}/\mathcal{Z}_0) > 14$, yielding posterior probabilities smaller than $7 \cdot 10^{-7}$ for all prior combinations. The total significance is about 6.1σ.

Finally, when all the available data is included, we find that the significance is about 7.8σ, and the evidences $\log(\mathcal{Z}/\mathcal{Z}_0) > 25$, giving posterior probabilities of H_0 smaller than 10^{-11} for all combinations of priors.

We observe that, for the first two cases in Tab. 2, the p-values corresponding to the observed values of Q^2 (which, again, are directly unrelated to the model probabilities) are rather small, while the posterior probabilities of H_0 are still significantly non-zero. This is
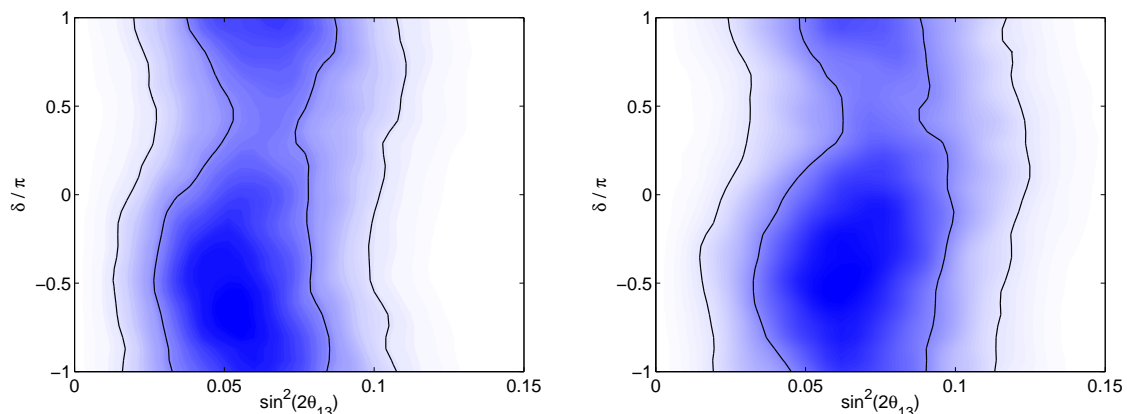
Figure 2. Two-dimensional posterior distributions for $\sin^2(2\theta_{13})$ and δ, using uniform priors on these variables and the data analyzed in Ref. [2]. Left (right) plots are for normal (inverted) neutrino mass ordering. The two black curves enclose 68 % and 95 % of the posterior probability, respectively.
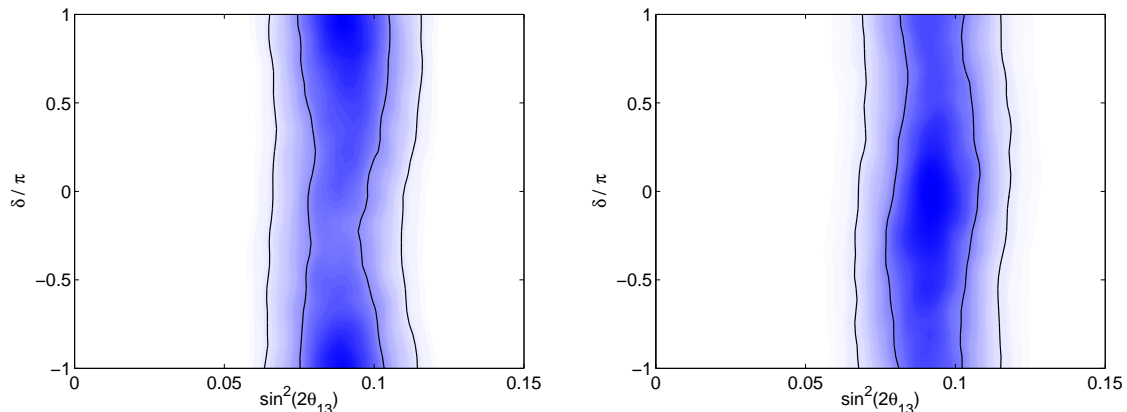

Figure 3. Same as Fig. 2, but using all available data.

the automatic Occam's razor effect of model selection at work: although the hypotheses with non-zero θ_{13} can accommodate significantly better fits, this is expected since these hypotheses have more freedom in their predictions, and this lack of predictivity is punished when using model selection. Although the maximum likelihoods are very large, the average ones are not, and these are what matters in model selection. Hence, the simpler model with $\theta_{13} = 0$, being more predictive, is not strongly disfavored. The situations are different for the three cases in the last three rows of Tab. 2. There, the likelihoods are so much larger for the alternative models that they are very strongly preferred, even though they are still less predictive and hence punished by the Occam's razor effect.

Since the posterior probability of H_0 is so incredibly small when including all data and in practice completely independent of any prior assumptions, we conclude that θ_{13} is
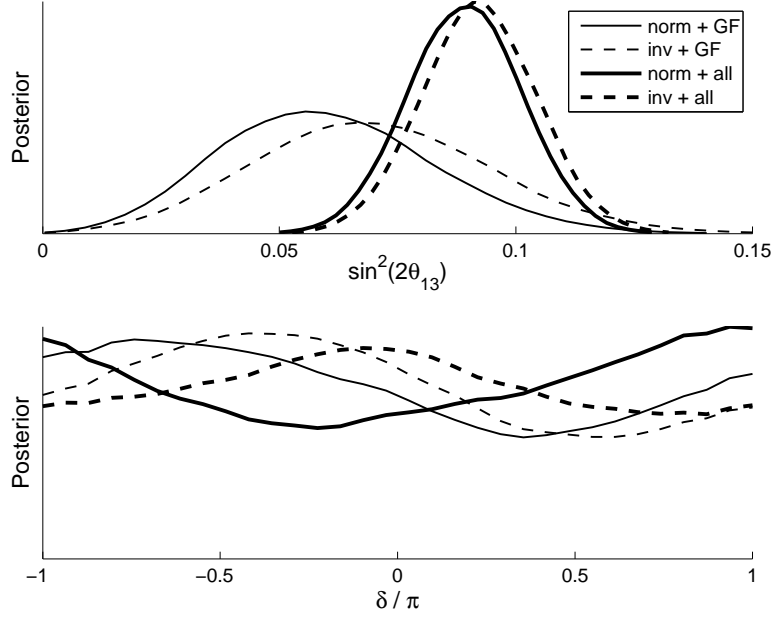
Figure 4. One-dimensional posterior distributions of $\sin^2(2\theta_{13})$ and δ for different combinations of mass orderings and data sets.

non-zero with a probability practically equal to one. The next question is then if there is evidence of CP-violation in neutrino oscillations, *i.e.*, a non-trivial value of the phase δ. Since the three models H_1, H_2, and H_3 all have θ_{13} as a free parameter, the resulting Bayes factors will to a very good approximation be independent of the prior on θ_{13}. Using all available data, we obtain the Bayes factors for normal mass ordering

$$\log(\mathcal{Z}_2/\mathcal{Z}_1) \simeq 0.5, \quad \log(\mathcal{Z}_3/\mathcal{Z}_1) \simeq 0.2, \tag{3.22}$$

while for the inverted ordering, the result is

$$\log(\mathcal{Z}_2/\mathcal{Z}_1) \simeq -0.2, \quad \log(\mathcal{Z}_3/\mathcal{Z}_1) \simeq -0.3. \tag{3.23}$$

Hence, there is evidence neither for nor against CP-violation in neutrino oscillations. This was of course expected since the likelihood depends very weakly on δ. If one assigns prior probabilities as in Sec. 3.2, *i.e.*, $\Pr(H_0) = 1/3, \Pr(H_1) = \Pr(H_2) = 1/6$, and $\Pr(H_3) = 1/3$, the posterior probability of H_3 is very close to 0.5 for both mass orderings $(|\Pr(H_3|\mathbf{D}) - 0.5| \lesssim 0.05)$, while the rest of the probability is shared between the H_1 and H_2 in slightly different proportions for the different mass orderings.

4 Summary and conclusions

We have described how to determine the evidence of a non-zero value of the leptonic mixing angle θ_{13} and a non-trivial CP-violating Dirac phase from neutrino oscillation data, using

Bayesian model selection. After having given a short, and hopefully clear, summary of the principles of Bayesian inference in general, and model selection in particular, we have applied it to the case of θ_{13} and δ.

We have compared the hypothesis $\theta_{13} = 0$ with hypotheses where θ_{13} can take on any value within its physical range and, by calculating the Bayesian evidences, obtained the evidence favoring a non-zero θ_{13}. After assigning prior probabilities to the different hypotheses, we have calculated their posterior probabilities. We have shown how recent data has step by step increased the evidence disfavoring $\theta_{13} = 0$, so that there is now overwhelming evidence of θ_{13} being non-zero.

Although the strong preference for a non-zero θ_{13} was expected, given the recent reactor data, this analysis also serves other purposes. Most importantly, we have described the Bayesian way of evaluating to what extent oscillation data supports the existence of CP-violation in neutrino oscillations through a non-trivial value of the phase δ. Since there is not much sensitivity to the value of δ in current data, we find, not surprisingly, no evidence neither for nor against CP-violation. However, in the future, when proposed experiments aiming to measure CP-violation have started taking data, this question will be of great importance. The method described here can be used to assess to which degree that future data favors CP-violation in neutrino oscillations, and act as an important complement to standard χ^2-analyses. In fact, it would be very interesting to evaluate how much proposed experiments would be able to provide (Bayesian) evidence of CP-violation (following, for example, Refs. [53–55]).

Finally, we note that Bayesian model selection could also be applied to other problems related to neutrino oscillations. For example, it could be used to decide which mass ordering of the neutrinos is preferred. To do this one would essentially need the full likelihood of all the oscillation parameters and make sure that one uses the same normalization of the likelihood throughout. Since future constraints on the phase δ is expected to depend very much on the assumed mass ordering, the best approach might be to define separate models for each assumption on δ and for each mass ordering, and then to perform model selection on all models simultaneously. In addition, one could use model selection to investigate whether θ_{23} is maximal or not, or if oscillation data requires the introduction of sterile neutrinos.

Acknowledgments

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